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DOUBLE-FLUID STEADY FILTRATION FLOWS IN NEAR-SHORE WATER-BEARING STRATA

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Two-dimensional stationary problems of filtration of a fluid having unknown contact (free) boundaries with fixed fluids of different density (water–air and salt and fresh waters) are studied. The paper considers various applied problems of this type, which are encountered, for example, in description of filtration in a water-bearing stratum of fresh water bordering on marine or salt ground waters: the problems of a fresh-water lens, a bottom-water cone near an imperfect well, equilibrium of two interfaces reaching drainage, etc. Unique solvability is proved for a wide class of contact problems of filtration of fluids of various density in porous channels with known parts of boundaries in the form of finite or infinite polygons.

Studies of filtration problems in a primal formulation relative to the parameters included in them (physical and geometric) reveal new qualitative properties of the solutions of these problems.

An algorithmic method for proving the solvability of the functional equations for such parameters of filtration flows with free (unknown) boundaries was first proposed in [1] and developed in [2–5].

1. Formulation of Contact Problems of Filtration Theory. The filtration equations are written as [6, p. 47]

$$\boldsymbol{v} = k\nabla\varphi, \quad \operatorname{div}\boldsymbol{v} = 0, \quad -\varphi = p/(\rho g) + x. \tag{1}$$

Here $\boldsymbol{v} = (u, v)$ is the filtration velocity vector, ($|\boldsymbol{v}|$ is the flow rate), $-\varphi$ is the hydraulic head (φ is the filtration potential), p and ρ are the fluid pressure and density, respectively, the gravitational vector $\boldsymbol{g} = (-g, 0)$ is in opposition to the Ox axis, which is perpendicular to the main fluid flow, and $k = \text{const} > 0$ is the filtration factor. The flow continuity equation $\operatorname{div}\boldsymbol{v} = 0$ in (1) allows one to introduce the stream function $\psi(x, y)$: $k\psi_y = u$ and $-k\psi_x = v$. Thus, two-dimensional steady filtration flows in homogeneous ($k = \text{const}$) porous media are described by the analytical function $w(z) = \varphi + i\psi$, which is the complex filtration potential ($z = x + iy$).

We consider the region $D = D(\rho_1)$ of the fluid filtration flow of density $\rho = \rho_1$ bounded by a specified finite or infinite polygon P and unknown curves of the free boundary L (water–air interface) and the interface Γ between fresh and salt waters. In turn, the polygon P includes regions P^k bordering on the fixed fluid of the same density $\rho = \rho_1$ with the condition $\varphi = \text{const}$ specified on them, impermeable regions P^j (confining beds), characterized by the condition $\psi = \text{const}$, and, in some problems, vertical lines of flow symmetry P^s: $y = \text{const}$, on which the condition $v = 0$ [$\boldsymbol{v} = (u, v)$] is specified.

For $z \in P^s$, we have $v = -k\psi_x = 0$, whence $\psi_x = 0$, and thus, P^s is a streamline $\psi = \text{const}$.

The free boundary L is a streamline $\psi = \text{const}$ with constant pressure on it ($p = \text{const}$), which leads to satisfaction of the condition $\varphi + x = \text{const}$. The interface Γ between fresh and salt waters is also a streamline $\psi = \text{const}$, on which the pressures $p_k = -g\rho_k(\varphi_k + x)$ ($k = 1, 2$) are identical, and this leads to the relation

$$\varphi - \lambda x = (\rho_2/\rho_1)\varphi_2 = \text{const}, \qquad \lambda = \rho_2/\rho_1 - 1 > 0, \qquad z \in \Gamma. \tag{2}$$

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In the plane of the complex potential $w = \varphi + i\psi$, the boundary conditions for $w(z)$ on $\partial D = P \cup L \cup \Gamma$ define the boundary of the region D^*, which consists of line segments $\varphi = \text{const}$ and $\psi = \text{const}$. In this case, depending on flow diagram, the inside angles $\gamma_k\pi$ at the vertices w_k of the polygon ∂D^* take one of the values: $\gamma_k\pi = \pi/2$, $3\pi/2$, or 2π at the finite vertices and $\gamma_k\pi = 0$ or $-\pi$ at the infinite vertices w_k.

2. Representation of Conformal Mappings. Problem of Parameters. We construct conformal mappings w: $E \to D^*$ and z: $E \to D$ of the upper half-plane E: $\operatorname{Im}\zeta > 0$ in the regions D^* and D. We assume that $z_k \in P$ ($k = \overline{0, n+1}$) are the vertices of the polygon P, t_k ($t_0 < t_1 < \ldots < t_{n+1}$) are the preimages of the vertices on the material axis ∂E [$z_k = z(t_k)$], $\alpha_k\pi$ are the inside angles at these vertices, and $l_k = |z_k - z_{k-1}|$ are the lengths of the finite links $P_k \subset P$ with the ends at the points z_k and z_{k-1} $\Big(P = \bigcup\limits_{k=1}^{n+1} P_k\Big)$.

From the condition $\varphi + x = \text{const}$ on the free boundary L, we obtain $dx/dt = -d\varphi/dt = |dw/dt|$. Similarly, from relations (2) on Γ, we have $dx/dt = \lambda^{-1}|dw/dt|$. In the problems studied below, the free boundary L is present only together with the interface Γ, and L and Γ reach horizontal drainage P^j: $x = \text{const}$, and on it, $dx/dt = 0$.

Thus, for $t\bar{\in}(t_0, t_{n+1})$, i.e., on the preimage of the interface $S = L \cup \Gamma \cup P^j$, the function dx/dt is known. Then, to determine the derivative $dz/d\zeta$, we have the boundary-value problem

$$\arg \frac{dz}{dt} = \delta_k\pi, \quad t \in [t_{k-1}, t_k]; \qquad \frac{dx}{dt} = q(t)\Big|\frac{dw}{dt}\Big|, \quad t\bar{\in}(t_0, t_{n+1}). \tag{3}$$

Here $\delta_k\pi$ is the angle between the link $P_k \subset P$ and the Ox axis, $q(t) = 1$ for $t \in L_*$, $q(t) = \lambda^{-1}$ for $t \in \Gamma_*$, and $q(t) = 0$ for $t \in P^j_*$; L_*, Γ_*, and P^j_* are the preimages of L, Γ and P^j, respectively [$L = z(L_*), \ldots$].

The canonical solution of the homogeneous problem (3) in the desired class of analytical functions is the derivative

$$\frac{dZ}{d\zeta} = C \prod_{k=0}^{n+1} (\zeta - t_k)^{\alpha_k - 1} \equiv C\Pi(\zeta) \qquad (C = \text{const}) \tag{4}$$

of the conformal mapping Z: $E \to D(\bar{P})$ of the upper half-plane E onto the region $D(\bar{P})$, bounded by the polygon $\bar{P} = P \cup P_0 \cup P_{n+2} = \bigcup\limits_{k=0}^{n+2} P_k$, where P_0 and P_{n+2} are infinite rays with the ends at the points z_0 and z_{n+1}. Writing the solution of the nonhomogeneous problem (3) in a standard manner via the solution of the homogeneous problem (4), we arrive at the following representations for the derivatives $dw/d\zeta$ and $dz/d\zeta$:

$$\frac{dw}{d\zeta} = K e^{i\beta\pi} \prod_k (\zeta - \tau_k)^{\gamma_k - 1} \equiv \Pi_0(\zeta), \qquad \frac{dz}{d\zeta} = \Pi(\zeta) M(\zeta),$$
$$\Pi(\zeta) = \prod_{k=0}^{n+1} (\zeta - t_k)^{\alpha_k - 1}, \qquad M(\zeta) = \frac{1}{\pi i} \int_{S_*} \frac{q(t)|\Pi_0(t)|}{\Pi(t)(t - \zeta)}\, dt \quad (S = z(S_*)). \tag{5}$$

Here τ_k are preimages of the vertices w_k of the polygon ∂D^*, which coincide with some of the parameters t_j.

Every vector $T = (t_1, \ldots, t_n)$ ($K = 1$, $t_0 = -1$, and $t_{n+1} = 1$ are fixed) substituted into (5) corresponds to a certain polygon $P(T)$ with links $P_k(T)$ parallel to $P_k \subset P$. The required constants t_k ($k = \overline{1, n}$) should be determined such that $P(T)$ coincides with the specified polygon P. We write the system of equations for t_k for the general case of the filtration region D, where on P there are two infinite vertices $z_s = \infty$ and $z_m = \infty$ ($0 \leqslant s < m \leqslant n+1$) upstream and downstream, respectively.

On each of the infinite links P_k and P_{k+1} ($k = s, m$) adjoining to the vertices $z_s = z_m = \infty$, we fix two different points and include them in the number of vertices P with vertex angles equal to π.

The required constants t_k ($k = \overline{1, n}$) are obtained from the system

$$l_k = \int_{t_{k-1}}^{t_k} |\Pi(t)M(t)|\, dt, \qquad k = \overline{1, n+1}, \quad k \neq s,\ s+1,\ m,\ m+1,$$
$$l_s + i l_{s+1} = \int_{t_0}^{t_{s+1}} \frac{dz}{d\zeta}\, d\zeta, \qquad l_m = \operatorname{Im} \int_{t_0}^{t_{m+1}} \frac{dz}{d\zeta}\, d\zeta. \tag{6}$$

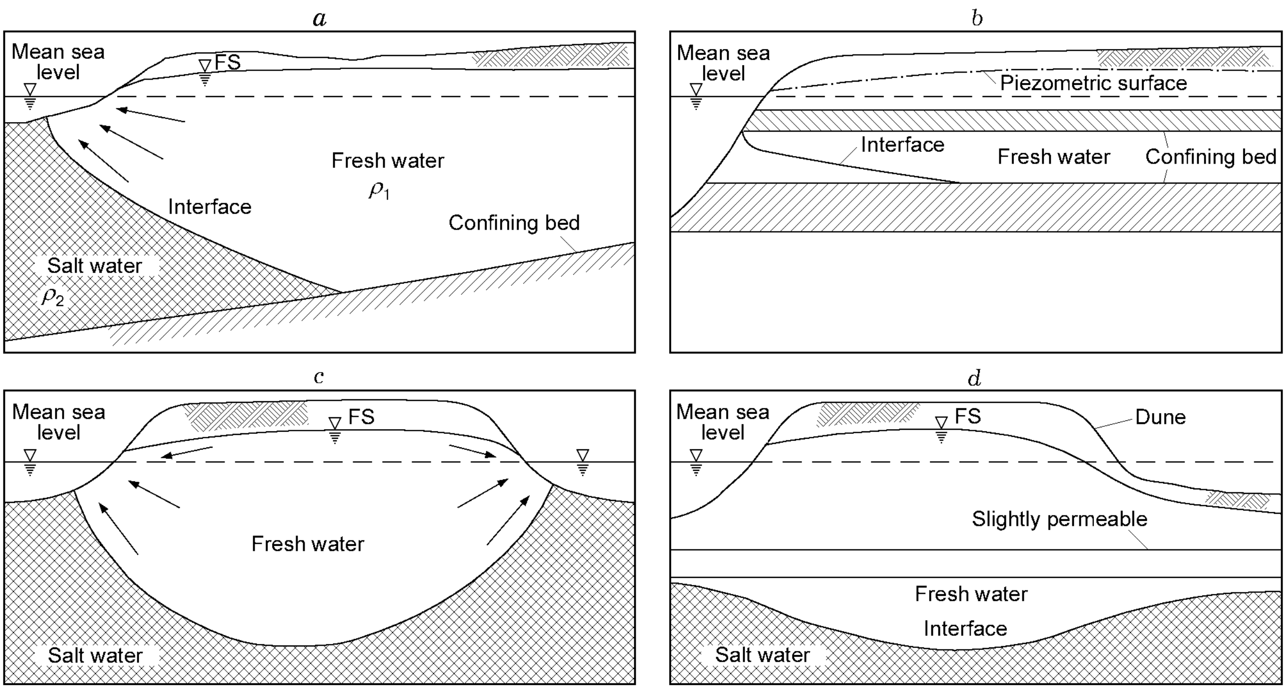

Fig. 1. Diagrams of near-shore flows (FS is the free surface): (a) free-flow horizon; (b) pressure horizon; (c) ocean island; (d) Dutch type.

Here $l_k = |z_k - z_{k-1}|$ are the specified lengths of the finite links $P_k \subset P$, $l_s + il_{s+1} = z_{s+1}$, $l_m = \text{Im } z_{m+1}$, and $z_0 = z(t_0) = 0$.

Any two of the last three equations of (6) can be replaced by relations for specified finite flow depths H_s and H_m in the neighborhood of z_s and z_m:

$$H_k = \pi\Big|\frac{dz}{d\zeta}(\zeta - t_k)\Big|_{\zeta=t_k}, \quad k = s, m.$$

Contact filtration problems are studied by the continuity method, which involves transition from primal problems, for which unique solvability of system (6) for parameters is known, to more complex problems by deformation of the polygonal boundaries of the region D [1, 2]. We consider a number of such initial problems, which are, in addition, of independent interest. Figure 1 [7, p. 287] shows interfaces of various shapes in near-shore water-bearing strata. When the interface intersects a confining bed, the region of salt water takes the shape of a wedge (Fig. 1a and b). Figure 1c and d shows a fresh-water lens floating on salt water.

3. Interface between fresh and salt waters under a dam. Figure 2a shows a diagram of fluid filtration in the region $D = D(\rho_1)$ under a narrow dam simulated by the line segment Ox in the presence of a fixed underlying bed $D(\rho_2)$ ($\rho_2 > \rho_1$) of salt groundwater [6, p. 333].

It is assumed that in the permeable regions $y > 0$, $y < 0$ of ∂D, the heads $\varphi = H/2$ and $\varphi = -H/2$, respectively, are specified [$\varphi \equiv \varphi_1$ in $D = D(\rho_1)$], and the unknown interface $\Gamma = D(\rho_1) \cap D(\rho_2)$ between the fresh and salt fluids is a streamline $\psi = 0$. The equality of pressures $p_k = -g\rho_k(\varphi_k + x)$ ($k = 1, 2$) on Γ leads to boundary condition (2).

For the derivatives of the conformal mappings w: $E \to D^*$ and z: $E \to D$, analogs of representations (5) hold:

$$\frac{dw}{d\zeta} = \frac{H}{\pi}(1-\zeta^2)^{-1/2}, \qquad \frac{dz}{d\zeta} = \lambda_0\Pi(\zeta)[M(\zeta) + iC],$$
$$\Pi(\zeta) = (1-\zeta^2)^{-1}, \qquad M(\zeta) = \frac{1}{\pi i}\int_{-1}^{1}\frac{(1-t^2)^{1/2}}{t-\zeta}\,dt, \qquad \lambda_0 = \frac{H}{\pi\lambda}. \tag{7}$$

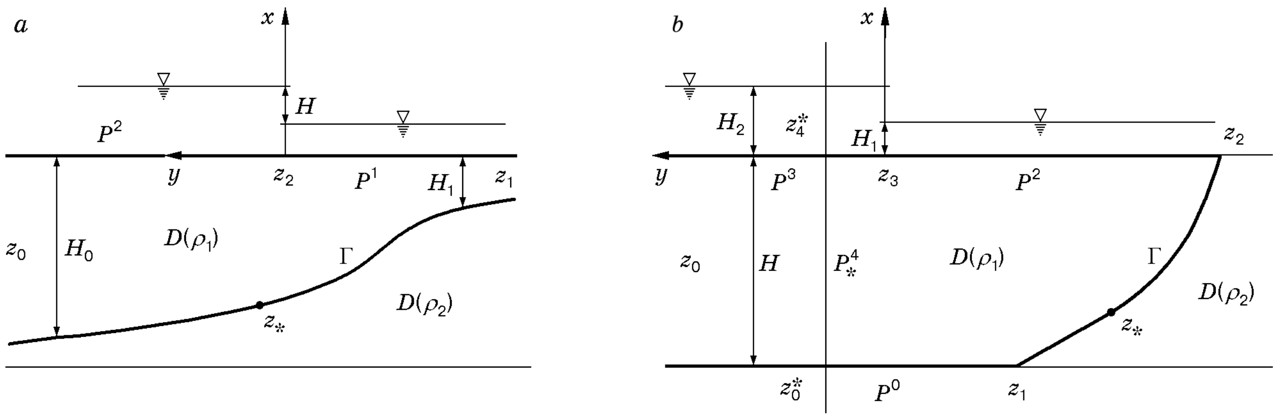

Fig. 2. Flow diagrams under a narrow dam (a) and in a water-bearing stratum (b).

The required material constant C should be determined from the equation for the specified depth H_0 in the lower pool $z_0 = \infty$:

$$\left|\frac{dz}{d\zeta}(\zeta+1)\right|_{\zeta=-1} = \frac{H_0}{\pi}.$$

We note that at $C = 0$, the derivative $dz/d\zeta$ in (7) has order of ζ^{-3} in the neighborhood of the point $\zeta = \infty$, and thus, the conformality of the mapping z: $E \to D$ is broken at this point.

The Cauchy-type integral $M(\zeta)$ in (7) satisfies the conditions $\operatorname{Re} M(t) = (1-t^2)^{1/2}$ ($|t| \leqslant 1$) and $\operatorname{Re} M(t) = 0$ ($|t| > 1$), to which, apparently, the function $M_0(\zeta) = (1-\zeta^2)^{1/2} - i(1+\zeta^2)^{1/2}$ also obeys. At $\zeta \to \infty$, the orders of functions $M(\zeta)$ and $M_0(\zeta)$ coincide, and, hence, $M(\zeta) = M_0(\zeta)$. Substituting the value of $M(\zeta)$ into (7), we obtain

$$\frac{dz}{d\zeta} = i\lambda_0(1-\zeta^2)^{-1}[N(\zeta)+C], \qquad N = -\frac{2}{\sqrt{\zeta^2-1}+\sqrt{\zeta^2+1}}. \tag{8}$$

Conservation of the orientation of the conformal mapping z: $E \to D$ and its nondegeneracy ($dz/dt \neq 0$, where $|t| < \infty$) lead to the condition $N(t) + C < 0$, $|t| > 1$, which is satisfied only for $C < -1$. With allowance for this, the constant $C = -(2\lambda H/H_1 + 1)$ is uniquely determined from the equation for H_0.

Remark 1. With the above normalization of the mapping z: $E \to D$, which is used in [6], the function $\Pi(\zeta)$ has a second order ζ^{-2} as $\zeta \to \infty$, which leads to a change of representation (5) for $dz/d\zeta$. Let now $t_1 = -1$ and $t_0 = 1$, and, thus, Γ is the preimage of Γ_*: $|t| > 1$. In this case, the representation $dw/d\zeta = (H/\pi)(1-\zeta^2)^{-1/2}$ remains unchanged, and $\Pi(\zeta)$ and $M(\zeta)$ are written in the form of (5):

$$\Pi(\zeta) = (\zeta-1)^{-1}, \qquad M(\zeta) = \frac{\lambda_0}{\pi i}\int\limits_{|t|>1}\left(\frac{t-1}{t+1}\right)^{1/2}\frac{dt}{t-\zeta}.$$

4. Interface in a Near-Shore Pressure Water-Bearing Stratum. A diagram of filtration is shown in Fig. 2b [7]. In the region $D = D(\rho_1)$ there is filtration of fresh water of density ρ_1, and the region $D(\rho_2)$ is occupied by fixed salt (marine) water of density $\rho_2 > \rho_1$. The rays

$$P^k = \{z \mid \operatorname{Re}(z-z_k) = 0,\ \operatorname{Im}(z-z_k) > 0\}, \quad k = 1,3 \quad (P^1 \equiv P^0)$$

are confining beds, i.e., streamlines $\psi = \psi_k = \text{const}$ ($\psi_0 = 0$ and $\psi_3 = Q$). On the unknown interface Γ, as in the problem considered in Sec. 3, the conditions $\psi = 0$, $\varphi = \lambda x + \varphi_*$, and $\lambda = \rho_2/\rho_1 - 1$ are satisfied The region $P^2 = \{z \mid x = 0,\ y_2 < y < 0\}$ of the seabed is an equipotential line $\varphi = 0$. From the conditions $\varphi = 0$ and $\varphi = \lambda x + \varphi_*$, at the point $z_2 = P^2 \cap \Gamma$ we obtain $\varphi_* = 0$. Then, at the point $z_1 \in \Gamma$, we have $w_1 = \varphi_1 = \lambda x_1 = -\lambda H$.

The derivatives of the conformal mappings w: $E \to D^*$ and z: $E \to D$ are represented in the form of (5), where $\Pi_0(\zeta) = K(\zeta-t_0)^{-1}[(\zeta-t_2)(\zeta-t_3)]^{-1/2}$ ($K > 0$), $\Pi(\zeta) = (\zeta-t_0)^{-1}$, and $S_* = \Gamma_*$ [$\Gamma = z(\Gamma_*)$]. The constants t_1, t_2, and $K = 1$ are fixed, and the required parameters t_0 and t_3 are determined from the equations

$$\lambda H = \int\limits_{\Gamma_*}|\Pi_0(t)|\,dt, \qquad l = |z_3 - z_2| = \int\limits_{t_2}^{t_3}|\Pi(t)|\,|M(t)|\,dt,$$

in which H and l are specified.

Having found t_0 and t_3, we calculate the flow rate Q: $Q = \pi|\Pi_0(\zeta)(\zeta - t_0)l|_{\zeta=t_0}$.

If the constant t_3 is also fixed, the constant t_0 can be found from the condition

$$H\lambda = \int_{\Gamma_*} |\Pi_0(t)|\, dt \equiv \Phi(t_0) \qquad \Bigl(\int_{\Gamma_*} = \int_{t_1}^{\infty} + \int_{-\infty}^{t_2} \Bigr). \tag{9}$$

In this case, the length $l = |z_3 - z_2|$ is not specified beforehand but is determined together with the flow rate Q after finding t_0. Let, for definiteness $t_2 = -1$, $t_3 = 0$, and $t_1 = 1$.

By the substitution of variables $t = -\tau$ in the integral over the ray $-\infty < t < t_2 = -1$, the function $\Phi(t_0)$ in (9) becomes

$$\Phi(t_0) = \int_1^{\infty} \frac{1}{\sqrt{t}} \Bigl(\frac{1}{(t - t_0)\sqrt{t+1}} - \frac{1}{(t + t_0)\sqrt{t-1}} \Bigr)\, dt \equiv \int_1^{\infty} \sigma(t_0, t)\, dt.$$

For $t_0 \in [0, 1]$, $\partial\sigma/\partial t_0 > 0$ and $\sigma(0, t) < 0$; $\sigma(t_0, t) \to +\infty$ as $t_0 \to 1$. Hence, there is a unique value of $t_0 \in (0, 1)$ for which relation (9) is satisfied.

5. Finite Water-Bearing Stratum. We consider the filtration diagram shown in Fig. 2b ($z_0^* = z_0$, $z_4^* = z_4$, and $P_*^4 = P^4$), where the water-bearing stratum, in contrast to the case studied in Sec. 4, has finite dimensions [8, p. 285].

On the feed contour $P^4 = \{z \mid -H < x < 0,\ y = y_0\}$ — the boundary between fixed and moving fresh waters — we set $\varphi|_{P^4} = \varphi_0 = 0$ ($\varphi_k = \operatorname{Re} w_k$, where $k = \overline{0, 4}$).

On the line $P^2 = \{z \mid x = 0,\ y_2 < y < 0\}$ of fresh water outflow — a horizontal filtration gap — we have $\varphi|_{P^2} = \varphi_2 = H_2 - \bar{H}_1$, where $\bar{H}_1 = H_1\rho_2/\rho_1$ is the potential of fresh water with respect to salt water, and, with allowance for the displacement scheme, $H_2 > \bar{H}_1$.

The confining beds P^0 and P^3 are streamlines $\psi = 0$ and $\psi = Q$. The unknown interface Γ between fresh and salt waters is characterized by the conditions $\psi = 0$ and $\varphi = \lambda x + \varphi_2$, whence it is necessary that $\varphi_1 = \varphi_2 - \lambda H > 0$.

Representations (5) for $dw/d\zeta$ and $dz/d\zeta$ become

$$\frac{dw}{d\zeta} = iK \prod_{k \neq 1} (\zeta - t_k)^{-1/2} \equiv \Pi_0(\zeta), \quad K > 0, \qquad \frac{dz}{d\zeta} = \Pi(\zeta) M(\zeta),$$
$$M(\zeta) = \frac{1}{\pi i} \int_{S_*} \frac{q(t)|\Pi_0(t)|}{\Pi(t)(t - \zeta)}\, dt, \tag{10}$$

where $S_* = \Gamma_*$ [$\Gamma = z(\Gamma_*)$], $\Pi(\zeta) = [(\zeta - t_0)(\zeta - t_4)]^{-1/2}$, and $q = \lambda^{-1}$. We fix the constants $t_0 = 0$ and $t_k = k - 1$ ($k = 2, 3, 4$), and obtain the unknowns parameters K and t_1 from the system

$$\varphi_k = \int_{t_0}^{t_k} |\Pi_0(t)|\, dt \qquad (k = 1, 2), \tag{11}$$

where φ_1 and φ_2 were calculated above.

In the integral for φ_1, we make the change of variables $t = st_1$ and write it as

$$K^{-1}\varphi_1 = t_1^{1/2} \int_0^1 s^{-1/2} \prod_{k=2}^{4} (t_k - st_1)^{-1/2}\, ds \equiv \varphi(t_1),$$

and from the second equation in (11), we calculate $K = [\varphi(1)]^{-1}\varphi_2$. Considering the ratio $\varphi_1/\varphi_2 = \varphi(t_1)/\varphi(1) \equiv \Phi(t_1)$, we note that $d\Phi/dt_1 > 0$ [$t_1 \in (0, 1)$], $\Phi(0) = 0$, and $\Phi(1) = \infty$. Hence, for all φ_1 and φ_2 ($0 < \varphi_1 < \varphi_2$) there is a unique value of $t_1 \in (0, 1)$ for which the equality $\varphi_1 = \varphi_2\Phi(t_1)$ is satisfied.

Thus, we established that system (11) is uniquely solvable for K and t_1.

The flow rate $Q = |w_4 - w_0|$ and the coordinates of all points $z_k \in \partial D$ ($k = \overline{0, 4}$) are determined from known values of t_k ($k = \overline{0, 4}$) and K.

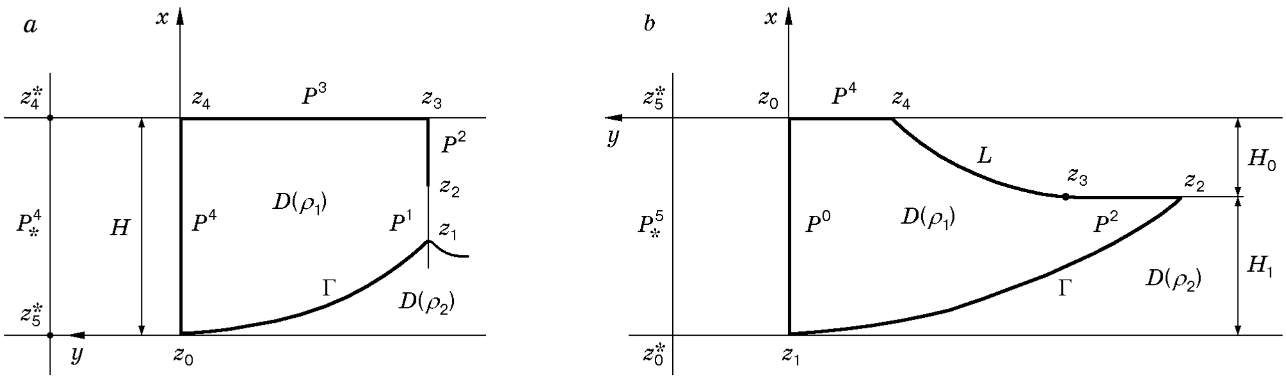

Fig. 3. Diagrams of filtration: (a) bottom-water cone; (b) fresh-water lens.

6. Bottom-Water Cone. Figure 3a shows a diagram of fresh water inflow into an imperfect well located symmetrically in a water-bearing stratum with underlying salt ground water [7; 9, p. 202].

On the feed contour $P^4 = \{z \mid y = 0,\ 0 < x < H\}$, we set $\varphi\Big|_{P^4} = 0$. The confining bed $P^3 = \{z \mid x = H,\ -l < y < 0\}$ is considered impermeable ($\psi\Big|_{P^3} = Q$). At the well $P^2 = \{z \mid x_2 < x < H,\ y = -l\}$ located on the symmetry axis $y = -l$ the following condition is specified:

$$\varphi\Big|_{P^2} = \varphi_2 > \varphi_1 = \lambda x_1 \qquad (\varphi_k = \operatorname{Re} w_k,\ k = \overline{0,4}).$$

In the unperforated region $P^1 = \{z \mid x_1 < x < x_2,\ y = -l\}$ of the symmetry axis, the projection v of the filtration velocity vector $\boldsymbol{v} = (u, v)$ is equal to zero ($\boldsymbol{v} = 0$), and, thus, $\psi = \text{const}$. The unknown interface Γ between moving fresh water and fixed salt water is a streamline $\psi\Big|_{\Gamma} = 0$, and on it the condition pressure equilibrium $\varphi\Big|_{\Gamma} = \lambda x$ is additionally satisfied. In this case, $\psi = 0$ over the entire line $(P^1 \cup \Gamma)$. The derivatives $dz/d\zeta$ and $dw/d\zeta$ are written in the form of (10), where

$$S_* = \Gamma_* \quad (\Gamma = z(\Gamma_*)), \qquad \Pi = \prod_k (\zeta - t_k)^{-1/2}(t_1 - \zeta)^{1/2} \quad (k = 0, 3, 4), \qquad q = \lambda^{-1}.$$

As in Sec. 5, the constants t_0, t_2, t_3, and t_4 are fixed, and the unknowns constants K and, t_1 are obtained from the uniquely solvable system (11).

Remark 2. Figure 3a shows a more general diagram of filtration of fresh and salt waters near an imperfect well, which is studied similarly to the diagram considered above ($z_4^* = z_4$, $z_5^* = z_5$, and $P_*^4 = P^4$).

7. Fresh-Water Lens. 7.1. *Symmetric Flow.* It is assumed that the interface between fresh and salt waters and the excess flow of fresh water directed toward the sea is in equilibrium, which leads to formation of a stable fresh-water lens floating on salt water [6, p. 334–338; 7]. The surface of the fresh-water flow to the sea is simulated by horizontal drainage. Half of this symmetric fresh-water flow in a near-shore water-bearing stratum is shown in Fig. 3b. Here L is the free boundary (water–air interface) on which the conditions $\psi = Q$ (Q is the required fresh water flow rate) and $\varphi + x = 0$ are satisfied, and Γ is the interface between fresh and salt waters characterized by the conditions $\psi = 0$ and $\varphi = \lambda x + \varphi_*$ ($\varphi_* = \text{const}$).

On the infiltration surface or at the bottom of the freshwater basin of small depth simulated by the segment $P^4 = \{z \mid x = 0,\ y_4 < y < 0\}$, we set $\varphi = 0$ and on the line $P^2 = \{z \mid x = -H_0,\ y_2 < y < y_3\}$ of fresh-water flow, $\varphi = H_0$ ($x = -H_0$ is the sea level). The line of symmetry $P^0 = \{z \mid -H_0 - H_1 < x < 0,\ y = 0\}$ is a streamline $\psi = 0$.

At the point $z_2 = \Gamma \cap P^2$, $H_0 = -\lambda H_0 + \varphi_*$, whence $\varphi_* = (1 + \lambda)H_0$. Then, at the point $z_1 = \Gamma \cap P^0$, we have $\varphi_1 = H_0 - \lambda H_1$ and $\varphi_0 = 0 < \varphi_1 < H_0 = \varphi_2$ ($\varphi_k = \operatorname{Re} w_k$). The last inequalities lead to the condition $H_1/H_0 < \lambda^{-1}$, which ensures motion of the fresh water toward the sea.

The derivatives $dw/d\zeta$ and $dz/d\zeta$ are written in the form of (10), where $\Pi = [(\zeta - t_0)(\zeta - t_1)]^{-1/2}$. The constants t_0, t_2, t_3, and t_4 are fixed, and K and t_1 are uniquely determined from system (11) (see Sec. 5).

A more general version of symmetric flow is shown schematically in Fig. 3b, were $z_4 = 0$, $z_5^* = z_5$, $z_0^* = z_0$, $P_*^5 = P^5$, and $P^0 = \{z \mid x = -H,\ y_1 < y < y_0\}$ ($H = H_0 + H_1$).

The segment $P^5 = \{z \mid -H < x < 0,\ y = y_5 > 0\}$ is a line of symmetry, and, thus, $\psi = 0$ on $P^0 \cup P^5$. On the feed line, $P^4 = \{z \mid x = 0,\ 0 < y < y_5\}$, we set $\varphi = 0$. In representation (10), we have

$$\frac{dw}{d\zeta} = \Pi_0(\zeta) = K\mathrm{e}^{i\beta\pi}\prod_{k=2}^{5}(\zeta - t_k)^{-1/2}, \qquad \Pi(\zeta) = [(\zeta - t_0)(\zeta - t_5)]^{-1/2}.$$

As in the previous case, the constants t_k, $k = \overline{2,5}$ $(-\infty < t_5 < t_0 < t_1 < \ldots < t_4 < \infty)$ are specified, and the parameters K and t_1 are uniquely determined from system (11), in which the integrals are calculated in the interval (t_5, t_k), $k = 1, 2$, and, therefore, the unknown parameter t_0 is not included in (11).

7.2. *General Case.* Figure 3b shows an asymmetric diagram of filtration of fresh water in a near-shore water-bearing stratum ($z_0^* = z_0$, $z_5^* = z_5$, and $P_*^5 = P^5$), where the polygon $P^4 \cup P^5$ is a feed contour $\varphi = 0$ or P^4: $\psi = Q$ and P^5: $\varphi = 0$. We consider, for example, the first version. In this case, in representation (10) we have

$$\frac{dw}{d\zeta} = K\mathrm{e}^{i\beta\pi}\prod_{k=2}^{4}(\zeta - t_k)^{-1/2}(\zeta - t_0)^{-1/2} \equiv \Pi_0(\zeta), \qquad \Pi(\zeta) = [(\zeta - t_0)(\zeta - t_5)]^{-1/2}.$$

We set $t_3 = -1$, $t_4 = 0$, $t_0 = 1$, and $t_2 = 2$ and the parameters K and t_1 are uniquely determined from system (11) (see Sec. 5).

Remark 3. In the general case of symmetric flow (see Sec. 7.1), the parameter $t_0 \in (t_5, t_1)$ remained unknown, and in the asymmetric diagram (see Sec. 7.2) the parameter $t_5 \in (0, 1)$ is unknown. These parameters can be determined, for example, from the equation $l = \displaystyle\int_{t_4}^{t_5}\Big|\frac{dz}{dt}\Big|\,dt$, in which the length $l = |z_5 - z_4| = y_5 > 0$ $(z_4 = 0)$ of the feed contour is specified. By construction, $|z_5 - z_0| = H_0 + H_1$ is fixed. Therefore, on the polygon $P = P^5 \cup P^4 \cup P^0$, which connects the ends $z_4 = L \cap P^4$ and $z_1 = \Gamma \cap P^0$ of the free boundaries L and Γ, only the coordinate $y_1 = \operatorname{Im} z_1$ remains unknown.

8. Polygonal Boundaries. In Secs. 3–7, we considered primal problems in which the specified segments of the boundary in the filtration region D consist of line segments (finite or infinite) parallel to the coordinate axes. In some of these problems, the parameters of the required conformal mapping z: ∂D were determined very simply owing to the monotony of the functionals defining the system of equations for these parameters. In other problems, the solvability of the system follows immediately from results of [4], where this problem was studied for more complex geometry of the boundary of the region D.

In further use of the continuity method to prove the solvability of general problems of gravity filtration, the problems studied in Secs. 3–7 will serve as initial problems, from which, filtration problems with complex geometry of the region D are obtained by polygonal deformation of specified segments of the boundaries ∂D. With this approach, in the problems considered in Secs. 3–7, the confining beds ($\psi = \mathrm{const}$) and the boundaries of water basins ($\varphi = \mathrm{const}$) can be considered polygons P^k (finite or infinite) with vertices at the points z_j^k and the angles $\alpha_j^k\pi$ at them.

9. Unique Solvability of Contact Problems. Let us consider the general contact problems of fluid filtration in porous channels formulated in Secs. 1 and 2 that can be obtained by polygonal deformation of confining beds and boundaries of water basins in the primal problems studied in Secs. 3–7. We note that under such deformation, finite polygons can become infinite ones [1, p. 165].

Investigation of contact problems of fluid filtration in regions D in which the specified part of boundaries is polygons $P \subset \partial D$ involves proving the solvability of system (6) for the parameters t_k $(k = \overline{1,n})$ of the conformal mappings z: $E \to D$.

The vector $p = (l, \alpha)$, where $l = (l_1, \ldots, l_{n+1})$ and $\alpha = (\alpha_0, \ldots, \alpha_{n+1})$, is called a *geometrical characteristic* of the polygon P because it completely defines the geometry of P.

The characteristic p satisfies the following conditions of a *prime* (nondegenerate) polygon [2]: $|\ln l_{k+1}| \leqslant \delta^{-1}$ $(0 < \delta \leqslant \alpha_k \leqslant 2$ and $k = \overline{0, n+1})$ and $|P_{ij}| \geqslant \delta$ $(|i - j| \geqslant 2)$. Here $P_{ij} \subset D$ is an arbitrary curve connecting the links $(P_i, P_j) \subset P$. The set of prime polygons P is denoted by $G = G(\delta)$ $[P \subset G$ and $p = (l, \alpha) \in G]$.

In the proof of the solvability of (6) for t_k $(k = \overline{1,n})$, the properties of the conformal mapping w: $E \to D^*$ also play an important role.

For the case where just one free boundary L or Γ is present, the unique solvability of Eq. (6) was established previously [2] for a wide class of problems (called filtration-type problems [2]) in which the derivative $dw/d\zeta$ depends only on fixed preimages t_0 and t_{n+1} of the ends of the free boundary $z_k = z(t_k)$ $(k = 0, n + 1)$.

For primal filtration problems for the parameters t_k ($k = \overline{1,n}$) with just one free boundary L or Γ formulated in Secs. 1 and 2, the unique solvability follows immediately from the results of [4]. Therefore, we consider filtration problems for fluids with two free boundaries L and Γ reaching horizontal drainage.

In the problems studied in Sec. 7, we deform the confining beds ($\psi = \text{const}$) and the boundaries of the water basins ($\varphi = \text{const}$), by replacing them by an arbitrary prime (nondegenerate) polygon $P \subset G(\delta)$. For definiteness, we consider the asymmetric problem from Sec. 7.2. We assume that z^k, where $k = \overline{0, n+1}$ ($z^0 = z_4 = 0$ and $z^{n+1} = z_1$), are the vertices and ends of the polygon P, $\alpha^k\pi$ are the vertex angles, $l_k = |z^k - z^{k-1}|$ are the lengths of the polygon links P_k, and $P = \bigcup\limits_{k=1}^{n+1} P_k$.

The parameters t_k ($k = \overline{2,4}$) and t_0 are fixed, and t_1 and $K > 0$ are determined from Eq. (11). We denote the preimages of the vertices $z^k \in P$ by $\tau_k \in [t_4, t_1]$ ($k = \overline{0, n+1}$) and note that $\tau_0 = t_4$, $\tau_{n+1} = t_1$, and one of the parameters $\tau_j = t_0$, $j \leqslant n$ (let $\tau_n = t_0$) are fixed. The vector $\tau = (\tau_1, \ldots, \tau_{n-1})$ of the unknown constants τ_k ($k = \overline{1, n-1}$) is obtained from the system

$$l_k = \int\limits_{\tau_{k-1}}^{\tau_k} \Big|\frac{dz}{dt}\Big|\, dt \equiv f_k(\tau, \alpha) \quad (k = \overline{1,n}), \qquad \alpha = (\alpha^0, \ldots, \alpha^{n+1}),$$

in which one of the equations (e.g., the equation for l_n) is a consequence of the remaining equations (see Remark 3).

We write the system of equations for τ_k ($k = \overline{1, n-1}$) as a functional equation for $\tau = (\tau_1, \ldots, \tau_{n-1})$:

$$l = f(\tau, \alpha), \qquad f = (f_1, \ldots, f_{n-1}). \tag{12}$$

In representations (5), the function $M(\zeta)$ has the form

$$M(\zeta) = \frac{1}{\pi i} \int\limits_{t_1}^{t_4} \frac{h(t)}{\Pi(t)(t - \zeta)}\, dt, \qquad h = q(t)|\Pi_0(t)|.$$

We note that the limits of integration t_1 and t_4 and all the parameters t_k ($k = \overline{0,4}$) and K included in $h(t)$ are fixed. Therefore, for the solution $\tau = (\tau_1, \ldots, \tau_{n-1})$ of Eq. (12) that corresponds to the prime polygon $P \subset G(\delta)$ as in [2], we establish the validity of the inclusion (*a priori* estimates):

$$\tau \in \Omega = \{\tau \mid \tau_{k+1} - \tau_k \geqslant \varepsilon(\delta) > 0,\ k = \overline{0,n}\}. \tag{13}$$

Based on estimates (13), in [2], the following properties of the vector $f(\tau, \alpha)$ are established:

$$f \in C^2[\Omega \times G], \qquad \Big|\frac{Df}{D\tau}\Big| \geqslant d(\varepsilon, \delta) > 0. \tag{14}$$

Here $Df/D\tau = \{f_{ij}\}$ and $f_{ij} = Df_i/D\tau_j$ ($i, j \in \overline{1, n-1}$).

Estimates (13) and (14) ensure the applicability of the continuity method [1–3], according to which, from the unique solvability of the primal problems in Sec. 7 it follows that Eq. (12) has a unique solution $\tau = (\tau_1, \ldots, \tau_{n-1})$ for an arbitrary prime polygon $P \subset G(\delta)$.

10. Analysis of Results. The classical problems of filtration theory considered in Secs. 3–7 have been studied by other researches, whose papers are cited in monographs [6–9]. As a rule, these problems were studied by the method of filtration hodograph, and the problem of parameters of conformal mappings corresponding to these problems $z\colon E \to D$ was solved using a semi-inverse approach: various values of these parameters were fixed, and from them flow filtration characteristics (head, length of a dam, dimensions of drainage regions, etc.) were calculated.

A solution of the primal problem of conformal mapping parameters in the theory of filtration of fluids with free boundaries was first obtained in [1] for the case of finite regions, and in [2], it was extended to infinite regions.

The main goal of the paper was to choose primal filtration problems such that by deformation of their specified boundaries in the class of polygons using the continuity method, it would be possible to prove the solvability of the general primal problems of filtration theory. Analysis of the parameters for some of these well-known problems reveals new qualitative properties of their solutions.

For the problem considered in Sec. 3, Polybarinova-Kochina [6] constructed an analog of formula (8) for the derivative of the conformal mapping $z\colon E \to D$, which includes an arbitrary material parameter. In Sec. 3, it is established that this parameter is not arbitrary and is uniquely determined such that the conformal mapping conserves orientation and is nondegenerate on the boundary ($dz/dt \neq 0$ at $|t| < \infty$).

In Secs. 6 and 7, we first formulated and solved the primal problems of a bottom-water cone and a fresh-water lens, which have been extensively studied previously using approximate models (see references in [6–9]).

We note some properties of the mapping $z\colon E \to D$ (see Secs. 6 and 7):

(a) in the preimage t_1 of the point z_1, the function $(dz/d\zeta)(\zeta - t_1)^{-1/2}$ has a logarithmic feature, i.e., at this point z_1, the free boundary Γ is not a Lyapunov curve (cf. with the example in [10, p. 172]);

(b) at the point t_2, the derivative $dz/d\zeta$ is bounded, and, thus, the flow symmetry axis $y = -l$ (see Sec.6) or $y = y_5$ (see Sec. 7.1) is the tangential to Γ at the point.

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